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On one method of approximation of reachability set for linear model of optimal control

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ABSTRACT. We consider linear model of optimal control and discuss one approach to approximation of its reachability set.

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1. Introduction

Optimal control theory began to take shape as a mathematical discipline in the 1950s. The motivation for its development were the actual problems of automatic control, satellite navigation, aircraft control, chemical engineering and a number of other engineering problems.

In applications of optimal control it is often arises [1, 2, 3, 5] the problem of exact or approximate description of the reachability set of the linear controlled model given by the recurrence relations

$$y(t+1) = A(t)y(t) + B(t)v(t) \ , \tag{1}$$

$$y(0) = y_0, \ v(t) \in V(t), \ t = 0,1,2,...,T-1 \ . \tag{2}$$

Here $y(t)$ is state m - vector , $v(t)$ is control s - vector, $A(t)$, $B(t)$ are given $m \times m$ and $m \times s$ matrices accordingly, $V(t)$ is subset of r - dimensional vector space R^r, T is natural number.

Pair of discrete functions $y(t), v(t)$ satisfying conditions (1), (2) is called the *process* and consisting the process functions $y(t), v(t)$ are called *trajectory* and *control* accordingly. If we go through all the processes $y(t), v(t)$, the end points $y(T)$ of the trajectories will fill a certain set Y in space R^m called the reachability set of the system (1), (2) at time T .

The issues of structure, description and approximation of the reachability set, when r - dimensional parallelepipeds are taken as regions of $V(t)$, are the subject of this work. In this sense, it is related to studies [1], [2]. The main attention is focused on the exact description of the reachability set by a finite system of linear inequalities and the simplest but unimprovable internal approximation of the reachability set by a maximal cube or parallelepiped.

Under the assumptions made, the set Y is a polyhedron. Like any polyhedron, it can be described by a finite or infinite system of linear inequalities. Depending on the method of description, the problem of inscribing the maximal cube into the set Y is posed and solved differently. If in the first case the problem is solved trivially, then in the second case it is equivalent to a multi-extremal problem of mathematical programming - maximization of a convex piecewise linear function on a polyhedron. Knowing the solution to the first problem allows us to specify a global solution to the second problem. Thus, a certain class of multi-extremal problems and a method for their global optimization are indicated, which is not reduced to the known schemes of cutting off [4] or a complete enumeration of the vertices of the polyhedron.

2. Specifying of the problem

Everywhere below we consider the regions $V(t)$ as r - dimensional parallelepipeds. Passing, if necessary, to recurrence relations in deviations and changing the scales along the coordinate axes, we can take $y_0 = 0$ and $V(t) = [-1,1]^s$ (standard s - dimensional cube).

Having successively performed recursion (1) for $t = 0,1,2,...,T-1$, we represent relations (1), (2) in the form

$$y(T) = \sum_{t=1}^{T-1} A(T-1)A(T-2)...A(t)B(t-1)v(t-1) + B(T-1)v(T-1),$$

$$(v(0), v(1),..., v(T-1)) \in V(0) \times V(1) \times ... \times V(T-1)$$

or in compact form

$$y = Ax, \quad x \in X, \tag{3}$$

where y is m- vector , x is n- vector, A is $m \times n$ matrix, $X = [-1,1]^n$ is standard cube, $n = sT$ is natural number. The notation (3) allows us to interpret the reachability set in terms of a linear transformation $A: R^n \to R^m$ given by the matrix A. According to this interpretation, $Y = AX$ is the result of a linear transformation of the cube X.

The subject of our attention will be two problems:

1. To describe the set Y analytically by a system of linear inequalities and
2. To construct the maximal cube inscribed in Y with edges parallel to the orthonormal basis vectors of the space R^m.

Next we assume $rank\ A = m,\ m < n$.

3. Algorithm of construction of reachability set

Introduce columns $a^1, a^2, ..., a^n$ of the matrix A and the sets Y_k of their linear combinations

$$y = x_1 a^1 + x_2 a^2 + ... + x_k a^k \ , \tag{4}$$

where $|x_1| \le 1,\ |x_2| \le 1,\ ... ,|x_k| \le 1,\ m \le k \le n$.

Comparing representations (3) and (4) of the points of the sets Y_k and Y, yields

$$Y_m \subset Y_{m+1} \subset ... \subset Y_n,\ Y_n = Y.$$

It is easy to see that the sets $Y_m,\ Y_{m+1},\ ... ,Y_n$ are polyhedrons formed by the intersection of a finite number of half-spaces. Indeed, if necessary, we renumber the columns of the matrix A so that the vectors $a^1, a^2, ..., a^m$ are linearly independent. Then formulas (4) for $k = n$ define a bijective mapping of the cube $[-1,1]^m$ into Y_m. Having written down the inverse bijection and taking into account inequalities (4), we obtain a description of Y_m by a system of linear inequalities

$$|b'_{im} y| \le 1,\ i = 1,2,...,m_1, \tag{5}$$

where $m_1 = m$, b'_{im} are the rows of the matrix $\left(a^1, a^2, ..., a^m\right)^{-1}$. From this it is clear that Y_m is a closed polyhedron symmetric with respect to the origin.

Assume that for $m \le k < n$ it is found polyhedron Y_k and

$$\left|b'_{ik}\, y\right| \le d_{ik},\ \ i = 1, 2, \dots, m_k \tag{6}$$

is its description. Comparing representations Y_k and Y_{k+1}, we see that Y_{k+1} is obtained by the union of all the shifts of the polyhedron Y_k by the vectors γa^{k+1}, $|\gamma| \le 1$. Or, in other words, the set Y_{k+1} is formed by the vectors $y + \gamma a^{k+1}$, where the variables y and γ independently run through the sets Y_k and $[-1,1]$, respectively.

It is clear that when Y_k is shifted by vectors γa^{k+1} the half-spaces (6) bounding Y_k will also shift in parallel; the maximum shifts correspond to $\gamma = \pm 1$. Due to this, the points $y \in Y_{k+1}$ will satisfy the inequalities

$$\left|b'_{i,k+1}\, y\right| \le d_{i,k+1},\ \ i = 1, 2, \dots, m_{k+1}, \tag{7}$$

where $b'_{i,k+1} = b'_{ik}$, $d_{i,k+1} = d_{ik} + \left|b'_{ik}\, a^{k+1}\right|$, $m_{k+1} = m_k$.

Additional $(m-1)$-faces (faces of dimension $m-1$) of Y_{k+1} will appear when shifting some $(m-2)$-faces of Y_k to γa^{k+1}. In order to describe the additional faces, we first establish three auxiliary results. The planes $\left|b'_{ik}\, y\right| \le d_{ik}$ from the description (6) of the polyhedron of Y_k will be called *bounding planes*.

Lemma 1. The $(m-2)$-face of a polyhedron Y_k is located in the intersection of the bounding planes $b'_1\, y = d_1$ and $b'_2\, y = d_2$ if and only if

1. $d_1 = \sum_{j=1}^{k} \left|b'_1\, a^j\right|$, $d_2 = \sum_{j=1}^{k} \left|b'_2\, a^j\right|$;
2. $\left(b'_1\, a^j\right)\left(b'_2\, a^j\right) \ge 0,\ \ j = 1, 2, \dots, k$;
3. $rank\left(a^j : b'_1\, a^j = b'_2\, a^j = 0\right) = m - 2$.

Proof of sufficiency of conditions 1-3 is evident. We show their necessity. Let the points $\bar{y}$ lay on the given bounding planes and on $(m-2)$-face of Y_k. Then for any point y from Y_k we have

$$d_1 = b'_1\, \bar{y} \ge b'_1\, y, \quad d_2 = b'_2\, \bar{y} \ge b'_2\, y.$$

Substituting y and $\overline{y}$ from (4) gives

$$\sum_{j=1}^{k}\left(x_j-\overline{x}_j\right)b'_1 a^j \le 0,\ \ \sum_{j=1}^{k}\left(x_j-\overline{x}_j\right)b'_2 a^j \le 0,$$

$$-1\le x_j,\overline{x}_j \le 1,\ \ j=1,2,...,k.$$

From here, in the virtue of arbitrary x_j, we conclude

(a) $\left|\overline{x}_j\right|\le 1$, $b'_2 a^j = 0$ if $b'_1 a^j = 0$;

(b) $\overline{x}_j = sign\left(b'_1 a^j\right)$, $\left(b'_1 a^j\right)\left(b'_2 a^j\right)\ge 0$ if $b'_1 a^j \ne 0$.

Consequently, for the points $\overline{y}$ it is true the following representation

$$y=\sum\nolimits_{j\in J}\overline{x}_j a^j+\sum\nolimits_{j\notin J} sign\left(b'_1 a^j\right)a^j,$$

where $J=\left\{1\le j\le k:\ b'_1 a^j = b'_2 a^j = 0\right\}$.

Obtained results along with the conditions (a) and (b) give the conclusions 1-3 of the lemma.

Lemma 2. Let L be the intersection of the planes $b'_1 y = d_1$ and $b'_2 y = d_2$. Then the union of all shifts of L by vectors γa, $|\gamma|\le 1$ lies in the plane

$$\left(\alpha_2 b_1-\alpha_1 b_2\right)' y=\alpha_2 d_1-\alpha_1 d_2,$$

where $\alpha_1 = a'b_1$, $\alpha_2 = a'b_2$.

If $\alpha_1=\alpha_2=0$ then the union of pointed above shifts coincides with L. The validity of the lemma is verified by direct calculations.

Lemma 3. The plane $b'y=d$ is a support one for a polyhedron Y if and only if

$$|d|=\sum_{j=1}^{k}\left|b'a^j\right|.$$

Proof follows from the definition of a support plane and a set Y_k.

Let us return to the description of the set Y_{k+1} and show how the system (7) is replenished by additional inequalities.

A. Choose indices i_1,i_2, $1\le i_1<i_2\le m_k$ and the corresponding planes bounding Y_k from the description (6). Let them be

$$b'_{i_1k}\, y = d_{i_1k}, \quad b'_{i_2k}\, y = d_{i_2k}. \tag{8}$$

Denote

$$b_1 = b'_{i_1k},\ b_2 = b'_{i_2k},\ d_1 = d_{i_1k},\ d_2 = d_{i_2k}.$$

B. Verify the conditions 1-3 of Lemma 1. If they are satisfied then the intersection of planes (8) contains the $(m-2)$-face of Y_k. Otherwise, go to step A.

C. Using Lemma 2, construct a plane

$$(\alpha_2 b_1 - \alpha_1 b_2)'\, y = \alpha_2 d_1 - \alpha_1 d_2, \tag{9}$$

$$\alpha_1 = a^{k+1} b_1,\ \alpha_2 = a^{k+1} b_2$$

containing all shifts of the intersection of planes (8) by vectors $\gamma a^{k+1},\ |\gamma| \le 1$.

D. Check the condition of supporting plane (9) to the polyhedron Y_k

$$|\alpha_2 d_1 - \alpha_1 d_2| = \sum_{j=1}^{k} \left|(\alpha_2 b_1 - \alpha_1 b_2)' a^j\right|.$$

If the equality is not satisfied, go to step A.

E. Replace m_{k+1} by $m_{k+1}+1$. Set

$$b_{m_{k+1},k+1} = \alpha_2 b_1 - \alpha_1 b_2,\ d_{m_{k+1},k+1} = |\alpha_2 d_1 - \alpha_1 d_2|$$

Thus, one more inequality is added to the system of inequalities (7).

We change the indices $i_1,\ i_2$ and proceed to step A. The algorithm ends when all non-repeating paired combinations of indices $1 \le i_1 < i_2 \le m_k$ are exhausted.

Remark 1. The modules in the description (7) appear due to the symmetry of Y_{k+1}.

Remark 2. Each pair of inequalities (6) generates four bounding planes. For reasons of symmetry, the intersections of only two pairs of planes are significant. Consequently, the total number of intersections of pairs of planes examined in the algorithm when moving from Y_k to Y_{k+1} is $2m_k(m_k - 1)$.

Sequential application of the algorithm for $k = m+1,\ m+2, \dots, n$ allows us to construct a system of inequalities

$$|b'_i\, y| \le d_i,\ i = 1, 2, \dots, M, \tag{10}$$

$$(b_i = b_{in},\ d_i = d_{in},\ M = m_n)$$

describing the set $Y = Y_n$ in the space R^m. As can be seen from the description, the set Y is a closed polyhedron symmetrical with respect to the origin.

4. Approximation of a reachability set

We consider the problem of inner approximation of a reachability set by cube (or parallelepiped). Introduce necessary notations. We denote by

$$Q(y,r) = \sum_{i=1}^{m} [y_i - r, y_i + r]$$

a cube with the center at a point $y \in R^m$ and the sides of the length $2r$ parallel to the coordinate axes. In particular, if $y = 0$ we regard $Q(0,r) = Q(r)$. For $c \in R^m$ we put

$$\|c\| = \sum_{i=1}^{m} |c_i| .$$

The following two results are immediately established.

Lemma 4. If $Q(y,r) \subset Y$, then $Q(r) \subset Y$.

Lemma 5. Inclusion $Q(r) \subset Y$ is equivalent inequalities

$$r\|b_i\| \le d_i,\ i = 1,2,...,M;\ r \ge 0.$$

From Lemma 1 and Lemma 2, it follows:

Corollary 1. Maximal cube inscribed into a reachability set has the center in the origin and the length of the side $2r*$ where

$$r* = \min_{1 \le i \le M} \frac{d_i}{\|b_i\|} \quad . \tag{11}$$

For parallelepiped

$$P(y,r) = \sum_{i=1}^{m} [y_i - r_i, y_i + r_i],$$

$$y = (y_1, y_2, ..., y_m),\ r = (r_1, r_2, ..., r_m) \ge 0,$$

the above results are generalizing as follows: if $P(y,r) \subset Y$, then $P(r) \subset Y$ and the latter is equivalent to the system

$$\sum_{j=1}^{m} \left| b_{ij} \right| r_j \le d_i, \; i = 1, 2, ..., M, \qquad (12)$$
$$r_j \ge 0, \; j = 1, 2, ..., m$$

in which b_{ij} is j-th component of the vector b_i. The maximum of a parallelepiped inscribed in Y can be understood in different senses: by volume, by the sum of its sides, or by the minimum side. Depending on this, the construction of the desired parallelepiped is reduced to a nonlinear programming problem with the target condition

$$r_1 \cdot r_2 \cdot ... \cdot r_m \to \max$$

or a linear programming problem with one of the conditions

$$r_1 + r_2 + ... + r_m \to \max;$$
$$r \to \max, \; (r \le r_1, \; r \le r_2, \; ... \; , \; r \le r_m)$$

under constraints (12) on the unknowns $r_1, \; r_2, \; ... \; , \; r_m$.

5. Associated multi-extreme problem

Previously, we used the description of the reachability set in the form of a system of linear inequalities. Let us move on to another description of Y - in the form of supporting half-spaces.

Each supporting for Y half-space with a normal vector $c \ne 0$ is defined by the inequality

$$c' y \le \left\| c' A \right\|, \qquad (13)$$

since

$$\max_{y \in Y} c' y = \sum_{j=1}^{m} \max_{|x_j| \le 1} c' a^j x_j = \sum_{j=1}^{m} \left| c' a^j \right| = \left\| c' A \right\|.$$

Then the polyhedron Y as a closed convex set is the intersection of all supporting half-spaces

$$Y = \left\{ y : c' y \le \left\| c' A \right\|, \; c \in R^m, \; c \ne 0 \right\}.$$

On the language of supporting half-spaces, the condition $Q(r) \subset Y$ is equivalent to the condition of belonging of $Q(r)$ to each supporting half-space (13), and this, in turn, is equivalent to the fulfillment of the continuum of inequalities

$$r\|c\| \le \|c'A\|, \ c \ne 0.$$

From here $r \le \dfrac{\|c'A\|}{\|c\|}$.

Consequently, the length r^* of the half of a side of maximum cube inscribed in Y is

$$r^* = \min_{c \ne 0} \frac{\|c'A\|}{\|c\|} = \min_{\|c\|=1} \|c'A\|. \tag{14}$$

Note that solution of extreme problem (14) exists by Weierstrass theorem and $r^* = 0$ if $rank\ A < m$.

We will call the extremal problem (14) an associated one. Let us consider it in more detail. By virtue of the definition of the norm, the associated problem consists of minimizing a convex piecewise linear function on a non-convex unit sphere - the boundary of a regular polyhedron. It is easy to see that this is a multi-extremal problem.

Theorem 1. Solutions c_s and c_v of the associated and auxiliary multi-extremal problems

$$\text{(s)} \qquad \|c'A\| \to \min, \ \|c\| = 1;$$

$$\text{(v)} \qquad \|c\| \to \max, \ \|c'A\| \le 1$$

exist and differ only in normalization

$$c_v = \frac{c_s}{\|c'_s A\|}, \quad c_s = \frac{c_v}{\|c'_v A\|}. \tag{15}$$

Proof. The solvability of the associated problem (s) is established above. By virtue of the assumption of the maximum rank of A, the set of admissible points in the auxiliary problem (v) is limited. Since this set is closed and the objective function is continuous, the auxiliary problem has a solution according to the Weierstrass theorem.

Let us prove formulas (15). Obviously, the points determined by (15) satisfy the constraints of the corresponding problems. Let c_s be a solution to the associated problem. Then, based on (14), we have

$$r^*\|c\| \le \|c'A\|$$

for any $c \neq 0$. In particular, if $\|c'A\| \le 1$ then from the previous inequality we obtain $r^*\|c\| \le 1$. At the same time, the first formula (15) gives $r^*\|c\| = 1$. This means that $\|c_v\| \ge \|c\|$ and c_v is a solution to the auxiliary problem.

Let us now assume that c_v is a solution to the auxiliary problem and c_s is found by the second formula (15). Then $\|c_v\| \ge \|c\|$ for $\|c'A\| = 1$ and by virtue of the positive homogeneity of conditions (v), we have $\|c'_v A\| = 1$. From this it follows

$$\frac{\|c'_v A\|}{\|c_v\|} \le \frac{\|c'A\|}{\|c\|}$$

or

$\|c'_s A\| \le \|c'A\|$ for $\|c\| = 1$. Theorem is proven.

We characterize the solutions of the auxiliary problems. Let $Q(r^*)$ be the maximal cube inscribed in Y. Then, in accordance with (11), the index set

$$I^* = \left\{ i:\ r^* = \frac{d_i}{\|b_i\|} \right\} \tag{16}$$

is defined. If $i \in I^*$, then

$$d_i = r^*\|b_i\| = \max_{y \in Q(r^*)} |b'_i y|. \tag{17}$$

Therefore, the planes $|b'_i y| = d_i$ corresponding to the inequalities (10) will be supporting to the cube $Q(r^*)$.

Theorem 2. The normal of any supporting plane to the cube $Q(r^*)$ from among the bounding planes of Y is a solution to the associated multi-extremal problem.

Proof. We fix any index i from I^*. By Lemma 3, in the virtue of property of supporting planes $|b'_i y| = d_i$ we have $d_i = \|b'_i A\|$. From here and (16) we determine

$$r^* = \frac{\|b'_i A\|}{\|b_i\|}$$

that taking into account (14) gives the result. Theorem is proven.

From Theorems 1 and 2, as a consequence, it follows that the normal of the bounding Y planes supporting to the cube $Q(r^*)$ determines the solutions of the auxiliary problem

$$\pm\frac{b_i}{\|b'_i A\|} \in \underset{\|c'A\|\le 1}{Arg\max}\|c\|,\ i \in I^*.$$

Conclusion

We discussed very important object of the theory of optimal control - the reachability set. We considered linear model and introduced one method of its construction in the form of the system of linear inequalities. Besides, for evaluation of the reachability set we offered its inner approximation by cube or parallelepiped of the maximum size. The latter is reduced to the problems of linear or nonlinear programming depending on objective function. In addition, we obtained special multi-extremal problem and its solution as the result of inner approximation of a reachability set defined in the form of supporting planes.

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